MATH 2850: INTRODUCTION TO LAPLACE TRANSFORMS

MOTIVATION: The Laplace Transform as a continuous analog of Power Series:

Suppose we have a convergent power series:

$$\sum_{n=0}^{\infty} a_n x^n = A(x).$$

The value of $A(x)$ is completely determined by the coefficients a_n. We may think of a_n as a real-valued function of the n; that is, $a_n = a(n)$ for $n = 0, 1, 2, ...$, so that we may think of a power series as a transform:

$$a(n) \rightsquigarrow A(x).$$

For example:

$$1 \rightsquigarrow \frac{1}{1-x}, \ -1 < x < 1,$$

since when $a(n) = 1$ for all n, we get the series $\sum_{n=0}^{\infty} x^n$. Similarly,

$$\frac{1}{n!} \rightsquigarrow e^x, \ -\infty < x < \infty,$$

since when $a(n) = \frac{1}{n!}$ for all n, we get the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

The sequence $a(n)$ is a discrete function; it's domain is restricted to just the whole numbers. Suppose we wished to find a continuous analog to the above process. It would be natural to start with:

$$\int_0^{\infty} f(t) x^t \, dt = F(x),$$

where we've replaced the discrete summation with the integral and the discrete function $a(n)$ with the continuous function $f(t)$. To ensure a reasonable chance that this integral converges, we restrict $0 < x < 1$, and we re-write:

$$x^t = e^{t \ln(x)},$$

to get:

$$\int_0^{\infty} f(t) e^{t \ln(x)} \, dt = F(x).$$

For the grand finale, we set $\ln(x) = -s$, so that $0 < s < \infty$. Hence, we get:

$$\int_0^{\infty} f(t) e^{-st} \, dt = F(s),$$

which takes a continuous function $f(t)$ and maps it to a function $F(s)$:

$$f(t) \rightsquigarrow F(s).$$

DEFINITION: Given a (piecewise) continuous function f on $[0, \infty)$, the **Laplace Transform** of f , $\mathcal{L}\{f(t)\}$ is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

NOTE: The variable t is integrated out and so we are left with a function of the parameter s, $F(s)$.

The Laplace Transform is hence defined for values s for which the integral converges.

EXAMPLE: Use the definition of Laplace Transform to derive formulas for the following:

- $f(t) = 1$

Ans: $\mathcal{L}\{1\} = \dfrac{1}{s}$, $s > 0$

- $f(t) = t$

Ans: $\mathcal{L}\{t\} = \dfrac{1}{s^2}$, $s > 0$

- $f(t) = t^2$

Ans: $\mathcal{L}\left\{t^2\right\} = \dfrac{2}{s^3}$, $s > 0$

- $f(t) = e^{4t}$

Ans: $\mathcal{L}\left\{e^{4t}\right\} = \dfrac{1}{s-4}$, $s > 4$

EXAMPLE: Derive a formula for $\mathcal{L}\{t^n\}$, assuming $n = 0, 1, 2, \ldots$

Ans: $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $s > 0$

EXAMPLE: Derive a formula for $\mathcal{L}\{e^{at}\}$, assuming $a \neq 0$

Ans: $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$

LINEARITY OF LAPLACE TRANSFORMS: If f and g have Laplace Transforms on a common domain:

$$\mathcal{L}\left\{c_1 f(t) \pm c_2 g(t)\right\} = c_1 \mathcal{L}\left\{f(t)\right\} \pm c_2 \mathcal{L}\left\{g(t)\right\} = c_1 F(s) \pm c_2 G(s)$$

EXAMPLE: Derive formulas for $\mathcal{L}\left\{\cosh(kt)\right\}$ and $\mathcal{L}\left\{\sinh(kt)\right\}$. Assume $k > 0$.

Ans: $\mathcal{L}\{\cosh(kt)\} = \dfrac{s}{s^2 - k^2}$ and $\mathcal{L}\{\sinh(kt)\} = \dfrac{k}{s^2 - k^2}$, $s > k$

EXAMPLE: Use $\cos(kt) = \cosh(ikt)$ and $\sin(kt) = \dfrac{1}{i}\sinh(ikt)$ to derive formulas for $\mathcal{L}\left\{\cos(kt)\right\}$ and $\mathcal{L}\left\{\sin(kt)\right\}$

Ans: $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2 + k^2}$ and $\mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2 + k^2}$, $s > 0$

EXAMPLE: Find the following Laplace Transforms:

- $\mathcal{L}\left\{3t^5\right\}$

Ans: $\mathcal{L}\left\{3t^5\right\} = \dfrac{3 \cdot 5!}{s^6} = \dfrac{360}{s^6}$, $s > 0$

- $\mathcal{L}\{3 - 4\sin(5t)\}$

Ans: $\mathcal{L}\{3 - 4\sin(5t)\} = \dfrac{3}{s} - \dfrac{20}{s^2 + 25}$, $s > 0$

- $\mathcal{L}\left\{e^{-3t}\cosh(2t)\right\}$

Ans: $\mathcal{L}\left\{e^{-3t}\cosh(2t)\right\} = \dfrac{1}{2(s+1)} + \dfrac{1}{2(s+5)}$, $s > -1$

FORWARD SHIFT: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to(s-a)} = F(s)_{s\to(s-a)} = F(s-a)$.

PROOF:

EXAMPLE: Find $\mathcal{L}\left\{e^{-3t}\cosh(2t)\right\}$ using the Forward Shift Theorem.

Ans: $\mathcal{L}\left\{e^{-3t}\cosh(2t)\right\} = \dfrac{s+3}{(s+3)^2-4} = \dfrac{s+3}{s^2+6s+5}$, $s >?$

EXAMPLE: Find the following Laplace Transforms:

- $\mathcal{L}\left\{t^2e^{3t}\right\}$

Ans: $\mathcal{L}\left\{t^2e^{3t}\right\} = \dfrac{2}{(s-3)^3}$

- $\mathcal{L}\{4e^{-t}\sin(2t)\}$

Ans: $\mathcal{L}\{4e^{-t}\sin(2t)\} = \dfrac{8}{(s+1)^2+4} = \dfrac{8}{s^2+2s+5}$

EXAMPLE: Find the Laplace Transform of $f(t) = \begin{cases} 0 & \text{if } t < 2, \\ e^{-t} & \text{if } t \geq 2 \ . \end{cases}$ using the definition.

Ans: $\mathcal{L}\left\{f(t)\right\} = \dfrac{e^{-2s-2}}{s+1}$

HOMEWORK: Pg. 403: 1-3, 5, 16*